OBJECTIFYING EARLY-NUMBER: A VISUAL NOMENCLATURE TO EXPRESS MATHEMATICAL DOMAIN KNOWLEDGE

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ABSTRACT

To address issues of divisive ideologies in the Mathematics Education community and to subsequently advance educational practice, an alternative theoretical framework and operational model is proposed which represents a consilience of constructivist learning theories whilst acknowledging the objective but improvable nature of domain knowledge. Based upon Popper's three-world model of knowledge, the proposed theory supports the differentiation and explicit modelling of both shared domain knowledge and idiosyncratic personal understanding using a visual nomenclature. The visual nomenclature embodies Piaget's notion of reflective abstraction and so may support an individual's experience-based transformation of personal understanding with regards to shared domain knowledge. Using the operational model and visual nomenclature, seminal literature regarding early-number counting and addition was analysed and described. Exemplars of the resultant visual artefacts demonstrate the proposed theory's viability as a tool with which to characterise the reflective abstraction-based organisation of a domain's shared knowledge. Utilising such a description of knowledge, future research needs to consider the refinement of the operational model and visual nomenclature to include the analysis, description and scaffolded transformation of personal understanding. A detailed model of knowledge and understanding may then underpin the future development of educational software tools such as computer-mediated teaching and learning environments.

Keywords: *early-number, knowledge, mathematics, modelling, reflective abstraction, understanding*

INTRODUCTION

Mathematics education literature has called for theorists and practitioners to abandon often divisive ontological and epistemological ideologies, and instead focus upon the synthesis of a consilience of theories that might form the basis for unifying and advancing contemporary practice (Goldin, 2003; Mayer, 2004). This paper reports upon the progress of an ongoing research activity aimed at responding to this challenge. Adopting an iterative design experiment-based methodology, a summary of the key literature that has shaped this response is firstly presented, followed by the proposition of an alternative theoretical framework and operational model. The operational model is then elaborated with further detail. In particular, a visual nomenclature is proposed that may be used to create detailed descriptions of domain knowledge. In the first iteration of the design experiment, the operational model and visual nomenclature was used to analyse and describe the domain of early-number counting, addition and subtraction. Excerpts from this activity are included in this paper to illustrate the various constructs of the visual nomenclature. Based upon this putting of theory into practice, the viability of the proposed theory as a tool to advance educational practice is briefly discussed, including the identification of activities that may be undertaken in future iterations of research.

LITERATURE REVIEW AND THEORETICAL PROPOSITION

Models of teaching and learning based on objectivist philosophical theories have had a significant influence on educational practice and are commonly referred to as traditional (Baroody & Dowker, 2003). Such theory and practice clearly delineate the roles of teacher and student, who may be thought of respectively as the transmitter and receiver of some fixed body of knowledge. Despite their wide influence, objectivist theory and practice have been widely criticised for often leading to impoverished, rote-like algorithmic understanding of mathematics (Chi & Bassok, 1989) characterised by an ability to solve only routine, previously experienced problems (Baroody, 2003; Hatano & Oura, 2003). At the heart of this criticism is the inadequacy of objectivist theory with regards to the recognition of a learner's past experience and how this might influence future learning (Lesh, 1985).

The advent of constructivism (based upon seminal theories of Piaget and Vygotsky) has led to attempts at reforming education and overcoming the limitations of objectivist-based practice. The constructivist paradigm has been perceived to reject objective reality in favour of experiential reality (Goldin, 2003), thus tending "to dismiss or deny the integrity of fundamental aspects of mathematical and scientific knowledge" (English, 2007, p. 120). Without an objective reference, important decisions regarding the design of instruction that will guide learning are impinged, especially when details of the construct and the processes by which it is formed are lacking (Lesh, Doer, Carmona, & Hjalmarson, 2003). Hence, whilst constructivist theory has served as a catalyst for educational reform, the development of constructivist educational practice has been significantly stymied (Baroody, 2003; Scardamalia & Bereiter, 2006; Simon, Tzur, Heinz, & Kinzel, 2004; Steffe, 2004). Thus, the current state of mathematics education has been characterised as being in the "clutches of constructivist ideologies" (English, 2007, p. 120).

To overcome the philosophical and practical difficulties facing mathematics education and research, Mayer (2004) has suggested that an ongoing challenge is to develop theory-driven practice which incorporates both learner-centred activity and teacher-guided promotion of cognitive activity. This would lead to the what Hatano and Oura (2003) described as adaptive expertise. Implicit in Mayer's call is the need for an objective model of domain knowledge upon which to base the promotion of cognitive activity. Mayer also challenged theorists and practitioners to move away from educational ideology, and to instead focus upon theory-driven research and in turn the development of science-based educational practice. Goldin (2003), who, whilst acknowledging the opinions of objectivists and constructivists, has made a similar call for new unifying theory: a consilience that recognises the kernels of truth within each of the ideologies.

Popper (1978) has provided a conceptualisation of knowledge which may integrate aspects of both objectivist and constructivist theory. Popper proposed the existence of three knowledge worlds, referred to as World 1, World 2 and World 3. These worlds respectively correspond to the world of actions; the world of mental thoughts; and the world of content of thought (that is, the organisation of ideas over which mental thoughts operate). Thus, the mental thoughts of World 2 mediate between the actions of World 1 and the organised ideas of World 3. Bereiter (2002) characterised World 3 knowledge as a tool to do real work, and that through experience the learner develops a relationship to that knowledge which is their World 2 understanding. Similarly, Woodruff (2005) described understanding as the manifold relationship that that the learner has to knowledge. Based upon Popper's model, Scardamalia and Bereiter's (2006) knowledge building paradigm promotes innovation, both in terms of the advancement of the learner's own personal knowledge frontier (i.e., their World 2 understanding) as well as the corporate knowledge frontier (i.e., the shared World 3 knowledge). They assert such innovation is essential to prepare learners for participation in the knowledge age of the 21st Century, a view which is consistent with Hatano and Oura's (2003) adaptive expertise.

Popper's three-world model of knowledge has provided a lens with which to synthesise and consilience of learning theories, including more traditional objectivism and more contemporary constructivism. This consilience is referred to as the *alternative theoretical framework*. In the following paragraphs, this framework is firstly outlined and is then complemented by the proposition of the *operational model* with which to put theory into practice.

The objective nature of World 3 knowledge proposed by Popper permits its explicit description, which may be thought of as a conceptual schema. It is claimed that such description will overcome a significant limitation generally attributed to constructivist theory. In keeping with social-constructivist theory, learning does not occur in isolation; it is mediated by the social milieu. Thus World 3 knowledge is explicitly describable and also discussable. It is also recognised that World 3 knowledge is not a fixed invariant truth, but is instead a product of innovation. Thus, World 3 knowledge may be described as the consensually developed knowledge of a domain shared by members of a (mathematical) community of practice. The alternative theoretical framework also supports the explicit description of a learner's World 2 understanding, or relationship, with respect to World 3 knowledge. Importantly, each learner can have their own unique relationship to the shared World 3 knowledge which reflects their past experience. Hence, the alternative theoretical framework also accounts for the idiosyncratic construction of understanding in the social milieu. The proposed alternative theoretical framework is summarised in Figure 1.

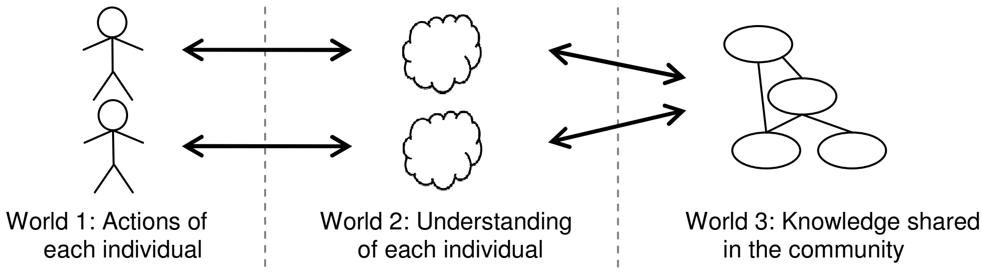

Figure 1. Summary of the alternative theoretical framework

The alternative theoretical framework has been proposed as a base which brings together aspects of various ideologies, in particular the recognition of the fundamental and improvable structures that define the domain of mathematics whilst recognising the significance of the unique learning experiences that characterise each learner's past and anticipated conceptual development. However, to add flesh to this theoretical framework, further learning theory is needed to explain an individual's conceptual development, and thus inform the proposition of the operational model that allows for the analysis and explicit description of World 3 knowledge and the description and explanation of World 2 understanding

Piaget considered the notion of reflective abstraction to be sufficiently powerful to describe the learner's entire development, as it " ranges over…all of the subject's cognitive activities [and] can be observed at every major stage of development " (Piaget, 1977/2001, p. 30). Reflective abstraction grew out of Piaget's theory of genetic epistemology, and was proposed as the mechanism by which accommodations in an individual's knowledge occur. It is based upon the identification and cognitive manipulation of properties of actions. That is, the reflection upon past experience and the construction of a conjunction that represents the inter-relationship between those actions (Dubinsky, 1991). Based upon a review of Piaget's various works, Dubinsky identified five forms of reflective abstraction: interiorisation, coordination, encapsulation, generalisation and reversal. Interiorisation is the transformation of a concept's expression from one form to another form which is more abstract, less detailed, and less contextualised (Olive, 2001; Steffe & Cobb, 1988), a notion that has been extended into the work of Pirie and Kieren (1994). Coordination is the "composition ... of two or more processes to construct a new one" (Dubinsky, 1991, p. 101) and is paralleled by the connection of each individual action into a coordinated whole (Piaget, 1977/2001). Encapsulation, or objectivising (Piaget, 1977/2001), involves the construction of new knowledge that brings together as one what were previously independent although related parts. This single object may represent the many parts of a complex process or the abstraction associated with a super-ordinate relationship. The transformations of coordination and encapsulation underpin the works of many neo-Piagetian theorists including Sfard (1991) and Gray and Tall (1994). Generalisation represents the application of existing processes and structures to a wider collection of problem phenomena (Dubinsky, 1991), and is evident in the work of Simon and Tzur et al. (Simon et al., 2004; Tzur & Simon, 2004). Finally, reversal was identified as the fifth transformation in which a new process is constructed that reverses or complements an existing process, which Piaget (1977/2001) discussed in terms of noticing the differences, not similarities, between actions.

In the operational model, reflective abstraction has been adopted as the primary mechanism by which to explain an individual's conceptual development and thus the transformation of World 2 understanding. A record of an individual's experience (actions) will be indicative of the individual's past conceptual development and their reflective abstraction-based transformation of World 2 understanding. Hence, the organisation of World 3 knowledge should also reflect the five transformations of reflective abstraction. Reflexively, the organisation of World 3 knowledge should suggest potential opportunities for a learner's transformation of World 2 understanding. Thus, reflective abstraction is basis for describing the organisation of World 3 knowledge and the analysis and anticipation of World 2 understanding. In the operational model, the description of World 3 knowledge represented diagrammatically using a visual nomenclature.

THE VISUAL NOMENCLATURE

In this section, the reflective abstraction-based visual nomenclature used to describe World 3 knowledge is introduced, including examples that demonstrate the nomenclature's use to model small fragments of knowledge in the domain of early-number mathematics

Using the visual nomenclature, some aspect of World 3 knowledge may be modelled or described using what is referred to as a *genetic decomposition*, a notion based upon the work of Dubinsky (1991). The genetic decomposition is a network-like structure composed of a collection of nodes and links. A particular genetic decomposition describes an aspect of a domain from some particular perspective, and so, as noted by Dubinsky, a genetic decomposition is not absolute since it is an interpretation of the knowledge shared in a community, constructed by one of its members.

The nodes in a genetic decomposition are referred to as *knowledge objects*. Each knowledge object is representative of some discrete aspect of domain knowledge to which the individual can develop an understanding. The links that connect knowledge objects are referred to as *associations*. Each association involves at least two knowledge objects and represents some proposition that spans these knowledge objects. In the following sub-sections, the different types of knowledge objects and associations are discussed in more detail, including the presentation of simple exemplar genetic decompositions that demonstrates the use of these constructs to describe small part of the early-number mathematics domain.

Knowledge Objects

The visual nomenclature defines three specific types of knowledge objects: *problems*, *concepts* and *representations*. These three objects are first introduced, and then their syntax in the visual nomenclature is presented.

At the core of learning activity is the solution of problematic situations, which is evident throughout learning theory and early-number literature, including problem classification taxonomies and the selection problem-solving strategies (e.g., Carpenter & Moser, 1983; Fuson, 1992; Tzur & Simon, 2004). The problem knowledge object is proposed to describe such problematic situations.

To solve problems, both procedural and conceptual knowledges are used, as discussed in various literature, including Gray and Tall (1994), Sfard (1991), Simon et al. (Simon & Tzur, 2004), Baroody (2003) and Bereiter (2002). Guided by this literature, including Baroody's comments regarding the integration of procedural and conceptual knowledge to better scaffold adaptive expertise, a single object type is used to describe specific instances of the procedural and conceptual knowledge of a domain.

A domain's language of discourse is comprised of various symbols or signs and is central to the articulation of mathematical knowledge (Fosnot & Dolk, 2001). The representation knowledge object is used to express each symbol or sign of the domain, including the concrete, iconic and symbolic forms discussed by various authors (e.g., Bruner, 1966; Payne & Rathmell, 1975).

In the visual nomenclature, the three different knowledge object types are respectively represented using rectangle, ellipse and parallelogram icons. The name of the knowledge object is used as a label for the icon. This notation is summarised in Figure 2.

Figure 2. Visual nomenclature constructs for describing problem, concept and representation knowledge objects

Knowledge Associations

Based upon a consideration of literature regarding reflective abstraction-based transformation of understanding, including Piaget (1977/2001) and Dubinsky (1991), six different yet related types of knowledge associations were defined as a part of the visual nomenclature: *inheritance*, *aggregation*, *solution*, *inversion*, *formalisation* and *expression*. In the following, each type is described in greater detail, including exemplars that demonstrate their use to construct simple genetic decompositions.

Inheritance Association
The inheritance association defines a collection of either problem, concept or representation knowledge objects that have a super-ordinate relationship with one another (Bruner, 1966): each *child* of the *parent* shares the characteristics of the parent, so each child is in some way similar or equivalent. That is, the inheritance association is derived from Piaget's transformation of encapsulation. In the visual nomenclature, the inheritance association is denoted using a solid line which connects the parent to its children, and is terminated at the parent end with an open triangle. In the case of two or more children, the solid line will branch and connect to each of the children. This syntax is illustrated in Figure 3, along with an exemplar of the construct's use.

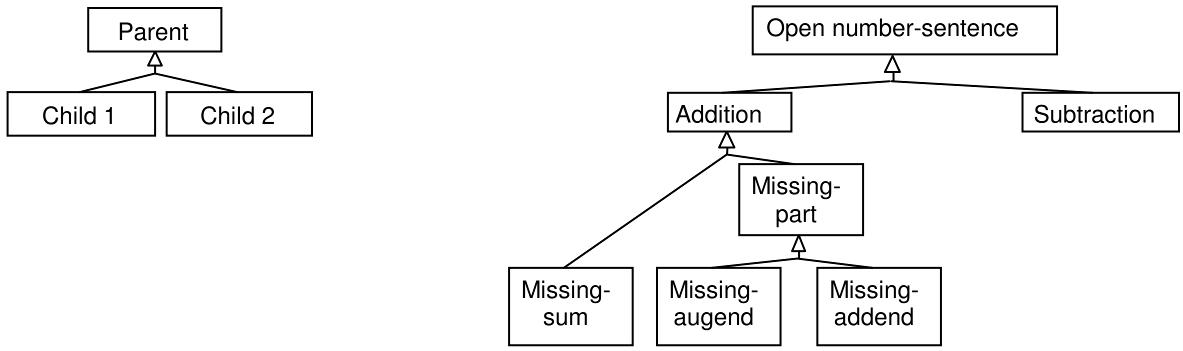

Figure 3. Definition and exemplar of the inheritance association syntax

Early-number addition and subtraction problems may be classified is by their underlying open number-sentence. The recursive partitioning of the set of all open number-sentences into the six specific open number-sentences can be described using the inheritance association to define a tree-like structure, as presented in Figure 3. At the first level of partitioning, the set of all open number-sentences (the parent) is split into two children: the set of addition problems and then set of subtraction problems. The addition problems are then split into those problems for which the whole is unknown (i.e., the missing-sum problem) and those problems for which one of the parts is unknown. These missing-part problems are then further partitioned into the missing-addend and missing-augend problems. Whilst not shown in Figure 3, the inheritance association could also be used to describe the partitioning of the subtraction problems.

Aggregation Association
Based upon the reflective abstraction transformations of coordination and encapsulation, as evidenced in the literature of Sfard (1991) and Gray and Tall (1994), the aggregation association describes the composition of several knowledge objects. That is, aggregation groups together *components*, or parts, of a more abstract *aggregate* whole. These parts may be either sub-problems of a larger problem, sub-concepts that together constitute a more complex concept, or several representations that can used together in a more expressive way. In the visual nomenclature, the aggregation association is denoted using a solid line which connects the aggregate to the components, and is terminated at the aggregate end with an open diamond. In the case of two or more components, the solid line will branch and connect to each of the components. This syntax is illustrated in Figure 4, along with an exemplar of the construct's use.

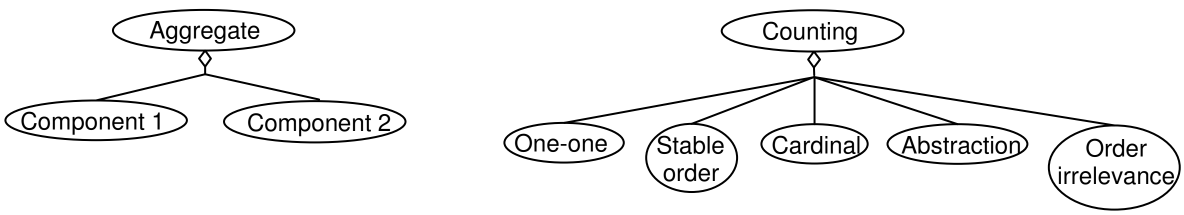

Figure 4. Definition and exemplar of the aggregation association syntax

Gelman and Gallistel (1978) identified five principles of counting: the one-one principle, the stable-order principle, the cardinal principle, the abstraction principle, and the order-irrelevance principle. These five principles can each be modelled in a genetic decomposition as five different concepts. Gelman and Gallistel asserted that when an individual uses these principles together in a coordinated fashion they demonstrate adult-like counting. In Figure 4, this coordination of parts is described using the aggregation association that defines counting to be the composition of the five different principles.

Solution Association

The solution association is used to link a problematic situation with a solution strategy, and has been influenced by several aspects of Piaget's reflective abstraction. A primary influence has been the transformation of coordination: the solution association identifies one (or more) concept(s) that may be coordinated together to solve an identified problem, and so in some ways the solution association is similar to aggregation. The solution association has also been influenced by the transformation of generalisation, in two related ways. Firstly, a particular concept may be used to solve a range of different problems. Secondly, a particular problem may be solved using a range of different concepts. This potential for problem-solving flexibility (an attribute of adaptive expertise) that comes from generalising a problem across a range of concepts or generalising a concept across a range of problems, can be described in a genetic decomposition using several different solution associations, each describing a different problem–strategy combination. The transformation of encapsulation has also influenced the solution association; Sfard (1991) and Gray and Tall (1994) discuss the encapsulation of a concept in terms of its coordination (or manipulation) with one or more other concepts to solve some problem. Thus, the inclusion of a particular concept along with one or more other concepts in a solution association describes the potential for that particular concept's encapsulation. In the visual nomenclature, the solution association is denoted using a solid line which connects the problem to the one or more concepts which are coordinated together to form the solution strategy. This solid line is terminated at the problem with a solid diamond, and in the case of two or more concepts the solid line will branch and connect to each of the concepts. This syntax is illustrated in Figure 5, along with an exemplar of the construct's use.

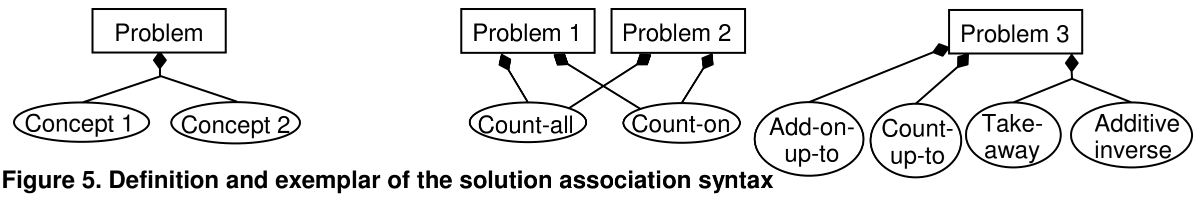

Figure 5. Definition and exemplar of the solution association syntax

Early-number literature (e.g., Fuson, 1992) has attempted to describe problem-strategy mapping, including the solution of addition and subtraction word problems. Fuson defined many types of word problems, including the following examples: Problem 1 – '*Connie had 5 marbles. Jim gave her 8 more marbles. How many marbles does Connie have altogether?*'; Problem 2 – '*Jim has 5 marbles. Connie has 8 more than Jim. How many marbles does Connie have?*'; and Problem 3 – *'Connie has 13 marbles. Jim has 5 marbles. How many marbles does Jim have to win to have as many marbles as Connie?'*.

Problems 1 and 2 are respectively examples of active and static problems (i.e., the problems stories either do or do not involve an action). The problems are also similar because they are both based upon the missing-sum open number-sentence (i.e., a + b = ?). Fuson proposed that missing-sum problems may be solved using the count-all strategy or the more advanced count-on strategy. These problem-strategy mappings are described in the genetic decomposition presented in Figure 5 using solution associations which link the two missing-sum problems to the two strategies. Problem 3 is based upon the missing-addend open number sentence (i.e., a+?=c), and may be solved using either of the additive add-on-up-to or (more advanced) count-up-to strategies. Like the missing-sum problems, this too is described using solution associations. Additionally, missing-addend problems may be solved using the subtractive take-away strategy, by coordinating it with the additive-inverse concept (i.e., that subtraction is the opposite of addition). In the genetic decomposition, this coordinated use of take-away and additive inverse is described using a solution association that includes both the concepts of take-away and additive-inverse.

Inversion Association

Based upon the transformation of reversal, the inversion association links together two knowledge objects, either problems or concepts, that are in some way complementary. The two knowledge objects are arbitrarily referred to as the *normal* and *complement* objects. Piaget asserted that the transformation of reversal is based upon the noticing of differences not similarities, and so the inversion association is often used to span knowledge objects that are also children of inheritance associations. In the visual nomenclature, the inversion association is denoted using a solid line which connects the two problems or two concepts, and which is terminated at one end with a solid circle and at the other with an open circle. This syntax is illustrated in Figure 6, along with an exemplar of the construct's use.

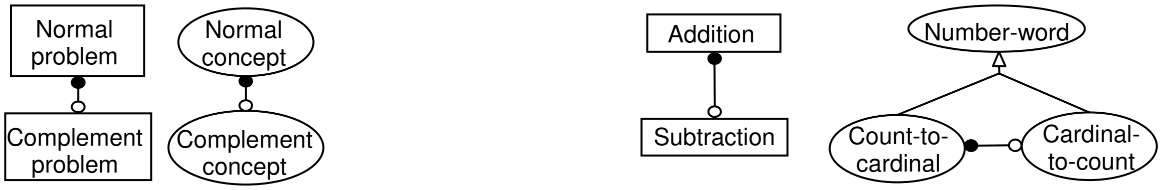

Figure 6. Definition and exemplar of the inversion association syntax

Consideration of early-number literature (e.g., Fuson, 1992) reveals numerous complementary problems or concepts, including problems of addition and subtraction, active and static problems, and procedural-like concepts of set-composition and set-decomposition, as well as less obvious pairings such as Fuson's count-to-cardinal and cardinal-to-count meanings of a number-word. The first and last of these examples are included in Figure 6 to demonstrate the inversion association's use, including the highlighting of the differences between the otherwise similar number-word meanings.

Formalisation Association

Based upon the transformation of interiorisation, the formalisation association describes the relative degree with which two representations embed contextual detail. The use of increasingly abstract signs and symbols of a domain is indicative of advancing conceptual development associated with a learner's interiorisation of some problem or concept. In the visual nomenclature, the formalisation association is denoted using a solid line which connects the two representations. This line is terminated at one end with an open arrowhead and at the other with a reversed open arrowhead, such that the arrowheads point in the direction of the abstraction. This syntax is illustrated in Figure 7, along with an exemplar of the construct's use.

Figure 7. Definition and exemplar of the formalisation association syntax

The formalisation association could be used to describe the association between Bruner's (1966) three distinct ways that a concept might be expressed, that is: action-based enactive (or concrete) representations; more abstract visual or sensory iconic representations; or context or action free symbolic representations. This organisation of representations is described by the genetic decomposition presented in Figure 7, in which two formalisation associations are used to show that iconic representations are more abstract than concrete representations, and that symbolic representations are more abstract than iconic representations.

Expression Association

In a domain of knowledge, the various problems and concepts may be expressed using the domain's set of signs and symbols. The expression association is used to describe the various ways by which a problem or concept may be represented. In the case that the various representations are also spanned by formalisation associations, then such an organisation of knowledge objects suggests the potential for interiorisation to occur. In the visual nomenclature, the expression association is denoted using a solid line which connects the problem or concept and the representation. This line is terminated at the problem or concept end with a solid arrowhead and at the representation end with reversed solid arrowhead, such that the arrowheads point towards the problem or concept expressed by the representation. This syntax is illustrated in Figure 8, along with an exemplar of the construct's use.

Figure 8. Definition and exemplar of the expression association syntax

Fuson (1992) identified a variety of strategies used to solve word-problems, including the add-on-up-to and count-up-to strategies which may be respectively differentiated by their expression using either perceptual (concrete or iconic) or conceptual (symbolic) representations. In Figure 8, this organisation of the domain is described using a combination of expression and formalisation associations. This example shows the two ways that the add-on-up-to concept can be expressed and hence the potential for the transformation of interiorisation to occur.

DISCUSSION

The proposed alternative theoretical framework and operational model, including the visual nomenclature, has been applied to the domain of early-number counting, addition and subtraction. To do this, a case-study based approach was adopted: seminal literature in the field of early-number (e.g., Carpenter & Moser, 1983; Fuson, 1992; Gelman & Gallistel, 1978; Olive, 2001; Steffe & Cobb, 1988) formed a set of case studies, each of which were analysed and described in detail using the operational model and visual nomenclature. In the case-study based activity, simpler genetic decompositions were firstly created, and then these were integrated together to form increasingly rich descriptions of the early-number domain.

In the preceding section, the constructs of the visual nomenclature were introduced and simple examples were provided to illustrate their use to describe the organisation problems, concepts and representations in the domain of early-number mathematics. In keeping with the approach taken in the research activity, a more complex genetic decomposition which integrates these examples is presented in Figure 9. Whilst only describing a small part of early-number counting, addition and subtraction, this genetic decomposition nevertheless exemplifies how rich descriptions of a domain of knowledge that embed the potential for reflective abstraction-based development of understanding may be created using the proposed visual nomenclature.

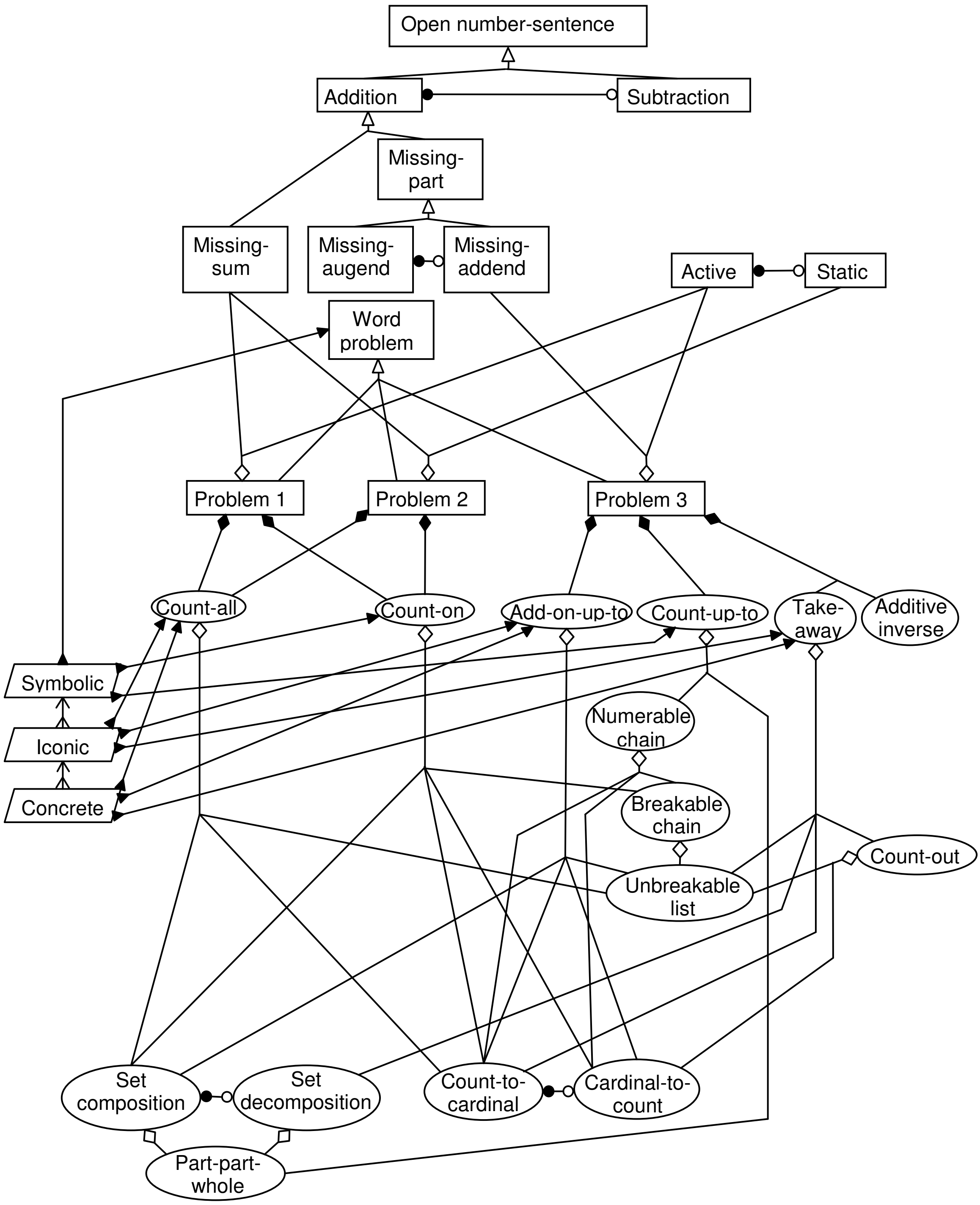

Figure 9. Genetic decomposition of early-number addition and subtraction problems, concepts and representations

The construction of genetic decompositions describing aspects of early-number World 3 knowledge, such as that presented in Figure 9, formed evidence with which to evaluate claims regarding the viability of the proposed theory. In the following sub-sections, several significant outcomes of this first iteration of research are presented, as well as the identification of activities that may be undertaken in future iterations of research.

Alternate Word Problem Classification Taxonomy

Literature regarding the semantic-based classification of early-number word problems (e.g., Carpenter & Moser, 1983; Fuson, 1992) tend to describe hierarchical taxonomies. Using the inheritance association, genetic decompositions descriptive of these taxonomies were constructed. Consideration of these genetic decompositions revealed that such hierarchies often introduced arbitrary distinctions between otherwise similar problems. Also, such semantic-based hierarchies cannot be reconciled with similarly hierarchical syntax-based taxonomies, such as open number-sentence based word-problem classification. To overcome these problems, an alternate heuristic was adopted that made use of both inheritance and aggregation associations to describe word-problems as aggregations of semantic and syntactic features. An alternate taxonomy that integrated both semantic and syntax-based classification was developed, an excerpt of which is included Figure 9: each of the three problems are described in terms of their component features. Thus, the proposed theory has been demonstrated as a viable tool with which to not only describe but also to advance the organisation of World 3 knowledge.

Composite Description of Early-number

Literature related to the classification of early-number counting, addition and subtraction concepts (Fuson, 1992; Gelman & Gallistel, 1978; Olive, 2001; Steffe & Cobb, 1988) was analysed and described using the operational model and visual nomenclature, and genetic decompositions were created for each. These genetic decompositions, which presented similar but not identical perspectives, were then integrated together to form a consolidated description of the domain's concepts, including problem solving strategies. This description together with the alternate word-problem classification taxonomy forms a composite description of the early-number counting, addition and subtraction domain that has not been presented elsewhere in literature.

However, the early-number literature that was analysed and described did not discuss the use of complex representations to express early-number problems and concepts (e.g., number lines). Future research activity should include the analysis and description of such literature, so as to extend the composite description of early-number counting, addition and subtraction, since such complex representations play a significant role in the scaffolding of a learner's development.

Visual Nomenclature Improvement

In this first iteration, the analysis and description of the early-number literature has given cause to exercise most constructs of the visual nomenclature. In doing this, one significant gap in the visual nomenclature has been identified. When solving some problem, the selection of a solution strategy is dependent upon the way in which the problem is expressed, and will in turn influence the way in which the solution strategy is expressed. Currently, the visual nomenclature only allows for the association of problem and solution; the influence of problem expression over concept selection is not modelled. The inclusion of this important information, perhaps by way of conditions attached to the solution association, must be addressed in future iterations of research.

Modelling of World 2 Understanding

The construction of the genetic decompositions of World 3 early-number knowledge has been necessary to evaluate the viability of the proposed visual nomenclature. Also, such descriptions will underpin future research activity. In the proposed alternative theoretical framework, understanding has been described as the individual's experience-based relationship to knowledge, thus the need for the description of World 3 knowledge upon which to base the consideration of World 2 understanding. Ongoing research activity involves the modelling of learner experiences and thus understanding.

This modelling or World 2 understanding is extending the visual nomenclature to include the *image* construct. Based upon the work of Pirie and Kieren (1994), the image construct groups together problem, concept and representation knowledge objects that define a particular learning experience (i.e., some problem solving activity). A chronologically-based sequence of such images, referred to as a *developmental trajectory*, may then be extracted. This developmental trajectory is representative of the learner's World 2 understanding. The developmental trajectory may be analysed to explain and anticipate the learner's conceptual development. Importantly, that analysis may be used to design learning activities that draw upon the learner's extant understanding to promote further conceptual development.

The explicit modelling of World 2 understanding with reference to World 3 knowledge may have particular relevance to the future design of computer mediated learning environments that respond to the learner in ways that embody constructivist theory. That is, environments which address the challenges of advancing constructivist teaching and learning practice, rather than environments which promote rote-like memorisation of facts.

CONCLUSION

Using the constructs of the visual nomenclature, a rich description of the early-number counting, addition and subtraction domain has been synthesised, excerpts of which have been included in this paper. Thus it is asserted that the viability of the proposed alternative theoretical framework and operational model, including the visual nomenclature, has been demonstrated. Whilst further elaboration and refinement of the operational model remains the subject of ongoing research activity, it is believed that the proposed theory may significantly advance teaching and learning practice, and thus address the aforementioned challenges presented in literature, including Baroody's (2003) call to not turn away from more complex theory and methods of instruction.

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